INSTITUTE OF MATHEMATICS, UNIVERSITY OF COLOGNE
12/11/2012

RISK PROCESSES WITH DEPENDENCE AND PREMIUM ADJUSTED TO SOLVENCY TARGETS

Ragnar Norberg, ISFA, Université de Lyon 1
Homepage: http://isfa.univ-lyon1.fr/∼norberg

Joint work with
Corina Constantinescu, University of Liverpool
Véronique Maume-Deschamps, ISFA, Université de Lyon 1

European Actuarial Journal 2012

ABSTRACT

This talk considers risk processes with various forms of dependence between waiting times and claim amounts. The standing assumption is that the increments of the claims process possess exponential moments so that variations of the Lundberg upper bound for the ruin probability are in reach. The traditional point of view in ruin theory is reversed: rather than studying the probability of ruin as a function of the initial reserve under fixed premium, the problem is to adjust the premium so as to obtain a given ruin probability (solvency requirement) for a given initial reserve (the financial capacity of the insurer). This programme is carried through in various models for the claims process, ranging from Cox processes with i.i.d. claim amounts to conditional Sparre-Andersen models. Special attention is given to a model for earthquake insurance risk, where dependence on the past is essential.

ON RUIN THEORY:

"Never in the field of scientific inquiry has so much effort been invested in the study of such a small probability"

An "academic subject ... too recherché to be of interest to the general run of actuaries"

"Risk theory based on stochastic processes is an indispensable tool in actuarial science and practice"

Origins, Present state, Developments, ... Applied Probability

"Ah – so it is the compensator!" – Applied Applied Probability

$(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$

Claims are marked point process, (T_i, Y_i), $i = 1, 2, \ldots$
T_i is time of occurrence of i-th claim and Y_i is its amount

Random measure N: $N(\mathcal{T} \times \mathcal{Y}) = \sum_i \varepsilon_{(T_i, Y_i)}(\mathcal{T} \times \mathcal{Y})$
For each $\mathcal{Y}$, $N(\mathcal{Y}) = (N([0, t] \times \mathcal{Y}))_{t \geq 0}$ adapted to $\mathbf{F}$

The $(\mathbf{F}, \mathbb{P})$-compensator of N is the random measure

$$\nu(dt, dy) = \mathbb{E}[N(dt, dy) \,|\, \mathcal{F}_t] = \lambda_t \, dt \, G_t(dy)$$

(choosing right-continuous version which is OK due to factor dt).

Conditional on $\mathcal{F}_t$,
λ_t is expected number of claims per time unit
G_t is the claim size distribution

Natural filtration of N is $\mathbf{F}^N$. The $(\mathbf{F}^N, \mathbb{P})$-compensator of N is

$$\nu^N(dt, dy) = \mathbb{E}\left[N(dt, dy) \,|\, \mathcal{F}_{t-}^N\right] = \mathbb{E}\left[\nu(dt, dy) \,|\, \mathcal{F}_t^N\right] = \lambda_t^N \, dt \, G_t^N(dy)$$

$\mathcal{F}_t^N$ represents the available information at time t

Short-hand $N_t = N([0, t], \mathbb{R})$. Claim number process $N(\mathbb{R}) = (N_t)_{t \geq 0}$ is simple counting process with $(\mathbf{F}, \mathbb{P})$-intensity λ_t and $(\mathbf{F}^N, \mathbb{P})$-intensity

$$\lambda_t^N = \mathbb{E}\left[\lambda_t \,|\, \mathcal{F}_t^N\right]$$

THE CASH FLOW PROCESS AND THE PROBABILITY OF RUIN

Premium rate process $(c_t)_{t\geq 0}$ is $\mathbf{F}^N$-adapted
Total cash flow process $X = (X_t)_{t\geq 0}$ is defined as

$$X_t = \sum_{i;\, T_i \leq t} Y_i - \int_0^t c_s \, ds = \int_0^t \int_0^\infty y \, N(ds, dy) - \int_0^t c_s \, ds$$

With initial reserve u the event of ruin is

$$\{\sup_{t\geq 0} X_t > u\} = \{T_u < \infty\}$$

$T_u = \inf\{t;\, X_t > u\}$ is the time of ruin – an $\mathbf{F}^N$ stopping time.

Probability of ruin is $\psi(u) = \mathbb{P}[T_u < \infty]$

CLASSICAL MODEL

$$\nu(dt, dy) = \lambda \, dt \, G(dy)$$

deterministic and time-independent λ and G.
Deterministic and time-independent premium rate

$$c = (1 + \rho) \, \lambda \, \mathbb{E}Y$$

where ρ is a positive (relative) safety loading:

$$\rho = \frac{c}{\lambda \, \mathbb{E}Y} - 1$$

Lundberg (1903): Moment generating function of G is denoted m_G. If there exists a positive *adjustment coefficient* R such that

$$c = \frac{\lambda}{R}\,(m_G(R) - 1)$$

then

$$\psi(u) \leq e^{-R\,u}$$

Cramér (1930): for large u the ruin probability is essentially exponential,

$$\lim_{u\to\infty} \left[\psi(u)\; e^{R\,u}\right] = \frac{\rho\,\mathbb{E}Y}{R\,\int_0^\infty y\,e^{R\,y}\,[1 - G(y)]\,dy}$$

DUBEY (1977)

$$\nu(dt, dy) = \Theta \, dt \, G(dy)$$

$$\Theta \sim \text{Gamma}(\alpha, \beta); \quad P_\Theta(d\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\,\theta} \, 1_{(0,\infty)}(\theta) \, d\theta$$

G is deterministic, independent of the time

$$c_t = (1+\rho) \, \lambda_t^N \, \mathbb{E}Y$$

$$\lambda_t = \Theta \text{ and (Bayes' formula) } \lambda_t^N = \mathbb{E}[\Theta | \mathcal{F}_t^N] = \mathbb{E}[\Theta | N_t] = \frac{N_t + \alpha}{t + \beta}$$

Dubey's result: $\psi(u)$ is the same as in the classical model.

A GENERALIZATION

Denote the $(\mathbf{F}^N, \mathbb{P})$-compensator of $(N_t)_{t\geq 0}$ by $\Lambda_t^N = \int_0^t \lambda_s^N \, ds$

Theorem 1
Assume that the Y_i are i.i.d. and independent of $(N_t)_{t\geq 0}$, that

$$\Lambda_t^N < \infty \text{ for } t < \infty \text{ and } \Lambda_\infty^N = \infty$$

and that, for each constant $\tau > 0$, the stopping time

$$\tilde{T}_\tau = \inf\{t; \Lambda_t^N \geq \tau\}$$

is bounded. Then, if

$$c_t = (1 + \rho)\, \lambda_t^N \, \mathbb{E}Y$$

with $\rho > 0$ constant, $\psi(u)$ is independent of the form of the intensity process λ^N, hence the same as in classical model.

Dubey's result is a special case:

$$\Lambda_t^N \geq \int_0^t \frac{\alpha}{s+\beta}\, ds = \alpha \ln\left(1 + \frac{t}{\beta}\right)$$

implying that $\tilde{T}_\tau \leq \beta\,(e^{\tau/\alpha} - 1)$.

PROOF

$\tilde{N}_\tau = N_{\tilde{T}_\tau}$ is adapted to $(\tilde{\mathbf{F}}, \mathbb{P}) = (\tilde{\mathcal{F}}_\tau)_{\tau \geq 0}$, $\tilde{\mathcal{F}}_\tau = F^N_{\tilde{T}_\tau}$.
Its compensator w.r.t. $(\tilde{\mathbf{F}}, \mathbb{P})$ is $\Lambda_{\tilde{T}_\tau} = \tau$ (optional stopping):
Since $N_t - \Lambda^N_t$ is $(\mathbf{F}^N, \mathbb{P})$-martingale, the martingale property carries over to bounded stopping times.
In particular, for any constants $\sigma < \tau$, $\tilde{T}_\sigma$ and $\tilde{T}_\tau$ are bounded, and so

$$\mathbb{E}[N_{\tilde{T}_\tau} - \Lambda_{\tilde{T}_\tau} | \mathcal{F}_{\tilde{T}_\sigma}] = N_{\tilde{T}_\sigma} - \Lambda_{\tilde{T}_\sigma}$$

This translates to

$$\mathbb{E}[\tilde{N}_\tau - \tau | \tilde{\mathcal{F}}_\sigma] = \tilde{N}_\sigma - \sigma$$

which means $(\tilde{N}_\tau - \tau)_{\tau \geq 0}$ is $(\tilde{\mathbf{F}}, \mathbb{P})$-martingale.

Since $\tilde{N}$ is simple counting process, it follows (Watanabe) that $\tilde{N}$ is Poisson process with intensity 1.
Finally, since $\tilde{T}_\tau$ ranges through $[0, \infty)$,

$$\begin{aligned}
\sup_{t\geq 0} X_t = \sup_{\tau\geq 0} X_{\tilde{T}_\tau} &= \sup_{\tau\geq 0} \left(\sum_{i=1}^{N_{\tilde{T}_\tau}} Y_i \, - \, (1+\rho) \int_0^{\tilde{T}_\tau} \lambda_s^N \, ds \, \mathbb{E}Y \right) \\
&= \sup_{\tau\geq 0} \left(\sum_{i=1}^{\tilde{N}_\tau} Y_i \, - \, (1+\rho) \, \tau \, \mathbb{E}Y \right) \quad \square
\end{aligned}$$

REMARK: Quite general path-dependent λ^N. Situations with dependence between counts and amounts are more difficult: cannot be carried back to the classical model with i.i.d. amounts.

USES OF THE PROBABILITY OF RUIN

$\psi(u)$ is function of initial reserve u. Can be seen as a measure of the riskiness of a given $X = (X_t)_{t \geq 0}$.
In the context of solvency control it tells us what u must be to satisfy

$$\psi(u) \leq \epsilon$$

In the classical model with a claim size distribution possessing exponential moments, Lundberg's upper bound can be used as a proxy for $\psi(u)$. Thus, the single number R is a measure of the riskiness of X. In particular, solvency requirement is met if

$$e^{-Ru} \leq \epsilon$$

Gives the easy answer $u = -\ln(\epsilon)/R$ to the question: what initial reserve is just sufficient to meet the solvency requirement?

TURN THE QUESTION AROUND:

For a given claims process and a given initial reserve, what premium rate is just sufficient to meet the solvency requirement?
Answer: adjust the premium so as to produce the pre-specified value

$$R = -\ln(\epsilon)/u$$

of the adjustment coefficient. This is the leitmotif of what follows.

ADJUST PREMIUM TO GIVEN (UPPER BOUND FOR) RUIN PROBY

Apply the idea above to more general models with non-homogeneity and dependence. Moment generating function of G_t^N is denoted $m_{G_t^N}$

First fix R in accordance with solvency requirement.
Next adjust premium dynamically so as to obtain Lundberg upper bound with this chosen R.

Theorem 2
Assume there exists an $R > 0$ such that $\mathbb{E}\exp[\int_0^t \int_0^\infty y\, N(ds, dy)] < \infty$ $\forall t$ ($m_{G_t^N}(R) < \infty$ $\forall t$ almost surely). If the premium rate is chosen as

$$c_t = \frac{\lambda_t^N}{R}\left(m_{G_t^N}(R) - 1\right) = \frac{\lambda_t^N}{R}\int_0^\infty \left(e^{R\,y} - 1\right) G_t^N(dy)\,,$$

then the Lundberg upper bound holds true.

(The result suggests a reason why the adjustment coefficient deserves its name):

PROOF: Define

$$M_t = e^{R\,X_t} = e^{R\left(\int_0^t \int_0^\infty y\,N(dt,dy) - \int_0^t c_s\,ds\right)}$$

The idea is to choose c such that M becomes a martingale. If this can be achieved, then we can recycle a well-known martingale proof due to Gerber.
Apply optional stopping to the martingale M and the bounded stopping time $T_u \wedge n$, $n > 0$ is a constant:

$$1 = M_0 = \mathbb{E}\left[e^{R\,X_{T_u \wedge n}}\right] \geq \mathbb{E}\left[e^{R\,X_{T_u \wedge n}}\,1[T_u \leq n]\right] \geq \mathbb{E}\left[e^{R\,u} 1[T_u \leq n]\right]$$

hence $1 \geq e^{R\,u}\,\mathbb{P}[T_u \leq n]$. Sending n to ∞, one obtains the Lundberg inequality.

Remains to prove that the premium function serves to martingalize M. Use Itô:

$$\begin{aligned} dM_t &= M_{t-}\left[\int_0^\infty \left(e^{Ry}-1\right) N(dt,dy) - R\,c_t\,dt\right] \\ &= M_{t-}\int_0^\infty \left(e^{Ry}-1\right)\left[N(dt,dy) - \lambda_t^N\,dt\,G_t^N(dy)\right] \\ &\quad + M_{t-}\left[\lambda_t^N \int_0^\infty \left(e^{Ry}-1\right) G_t^N(dy) - R\,c_t\right] dt \end{aligned}$$

M is martingale if the drift term is null. This is obtained by choosing c_t as in the Theorem.

Instantaneous safety loading at time t is

$$\rho_t^N = \frac{c_t}{\lambda_t^N \, \mathbb{E}_t^N Y} - 1$$

With the risk-adjusted premium rate in the theorem

$$\rho_t^N = \frac{\int_0^\infty \left(e^{R\,y} - 1 - R\,y\right) \, G_t^N(dy)}{R \int_0^\infty y \, G_t^N(dy)}$$

dependent on the claim amount distribution, but not on the intensity. Taylors' formula with reminder term says

$$e^{R\,y} = 1 + R\,y + \frac{1}{2} e^{h(R\,y)} \, R^2 \, y^2$$

where $0 \leq h(R\,y) \leq R\,y$. Thus

$$\rho_t^N = \frac{R}{2} \frac{\int_0^\infty e^{h(R\,y)} y^2 \, G_t^N(dy)}{\int_0^\infty y \, G_t^N(dy)} > 0$$

SPECIAL CASES

In order to determine

$$c_t = \frac{\lambda_t^N}{R} \int_0^\infty \left(e^{R\,y} - 1\right) G_t^N(dy)$$

we need to calculate $\nu^N(dt, dy)$, which goes by the tower rule

$$\nu^N(dt, dy) = \mathbb{E}\left[N(dt, dy) \,|\, \mathcal{F}_{t-}^N\right] = \mathbb{E}\left[\nu(dt, dy) \,|\, \mathcal{F}_{t-}^N\right] = \lambda_t^N \, dt \, G_t^N(dy)$$

The labour incurred depends on the model, which is determined by the form of $\nu(dt, dy)$

The simplest situation is when ν is deterministic.

When G is deterministic and independent of time, the risk-adjusted premium rate is

$$c_t = \lambda_t^N \frac{m_G(R) - 1}{R}$$

and ρ is constant. Theorem 1 tells us that we are essentially in the classical model, only using a different clock.

From these observations we conclude: in quest for a model that is genuinely beyond the classical and its equivalents, we need at least to introduce some kind of dependence between the claim amounts. One can create any conceivable dependence between counts and amounts by letting λ_t^N and G_t^N depend on the history $\mathcal{F}_t^N$.

We shall generate dependence by placing a probability distribution on parameters of well known processes (two-stage models or doubly-stochastic processes).

Introduce random element Θ representing unobservable risk factors
Assume $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ is of the form $\mathcal{F}_t = \mathcal{F}^{\Theta} \vee \mathcal{F}_t^N$.

The $(\mathbf{F}, \mathbb{P})$-compensator ν (or λ_t and G_t) is a function of Θ and possibly of the past claims history.

Θ may represent model uncertainty, but it may also represent fluctuating unobservable risk conditions.

PARAMETRIC MODELS

When Θ is finite-dimensional, then λ_t^{Θ} and G_t^{Θ} are parametric functions

Assume $G_t^{\Theta}(dy) = g_t^{\Theta}(y)\, dy$ (just to ease notation)

For a given full specification of the past history, use the binary nature of $N(dt, dy)$ to write

$$
\begin{aligned}
\nu^N(dt,dy) &= \mathbb{P}[N(dt,dy)=1 \,|\, N_t = n,\ N(dt_i,dy_i)=1, i=1,\dots,n] \\
&= \frac{\mathbb{P}[N_t = n,\ N(dt_i,dy_i)=1, i=1,\dots,n,\ N(dt,dy)=1]}{\mathbb{P}[N_t = n,\ N(dt_i,dy_i)=1, i=1,\dots,n]} \\
&= \frac{\int \mathbb{P}[N_t = n,\ N(dt_i,dy_i)=1, i=1,\dots,n,\ N(dt,dy)=1 \,|\, \Theta=\theta]\, P_\Theta(d\theta)}{\int \mathbb{P}[N_t = n,\ N(dt_i,dy_i)=1, i=1,\dots,n \,|\, \Theta=\theta]\, P_\Theta(d\theta)} \\
&= \frac{\int e^{-\int_0^t \lambda_s^\theta\, ds} \left[\prod_{i=1}^n \lambda_{t_i}^\theta\, g_{t_i}^\theta(y_i)\right] \lambda_t^\theta\, g_t^\theta(y)\, P_\Theta(d\theta)}{\int e^{-\int_0^t \lambda_s^\theta\, ds} \left[\prod_{i=1}^n \lambda_{t_i}^\theta\, g_{t_i}^\theta(y_i)\right] P_\Theta(d\theta)}\, dt\, dy
\end{aligned}
$$

EXAMPLE

$\lambda_t^\theta = \theta$ (homogeneous Poisson process)
$g_t^\theta(y) = \theta\, e^{-\theta\, y}\, 1_{(0,\infty)}(y)$ (exponential distribution).
$\Theta \sim$ Gamma(α, β).

$$\nu^N(dt, dy) = \frac{(2n+1+\alpha)(2n+\alpha)}{(t+\sum_{i=1}^n y_i + \beta)^2} \left(\frac{t+\sum_{i=1}^n y_i + \beta}{t+\sum_{i=1}^n y_i + \beta + y} \right)^{2n+2+\alpha} dt\, dy$$

$$\lambda_t^N = \frac{2\, N_t + \alpha}{t + \sum_{i=1}^{N_t} Y_i + \beta}$$

$$g_t^N(y) = \frac{2\, N_t + 1 + \alpha}{t + \sum_{i=1}^{N_t} Y_i + \beta} \left(\frac{t + \sum_{i=1}^{N_t} Y_i + \beta}{t + \sum_{i=1}^{N_t} Y_i + \beta + y} \right)^{2N_t+2+\alpha}$$

The claim size distribution does not possess positive exponential moments. (The infinitely large claims again.)

To remedy the problem, dock the left tail of the gamma distribution at $z > 0$ and use

$$P_{\ominus}(d\theta) = \frac{1}{I_{\alpha,\beta}(z)}\,\theta^{\alpha-1}\,e^{-\beta\,\theta}\,1_{(z,\infty)}(\theta)\,d\theta$$

$$\nu^N(dt, dy) = \frac{\int_z^\infty \theta^{2n+2+\alpha-1}\, e^{-(t+\sum_{i=1}^n y_i + y + \beta)\,\theta}\, d\theta}{\int_z^\infty \theta^{2n+\alpha-1}\, e^{-(t+\sum_{i=1}^n y_i + \beta)\,\theta}\, d\theta}\, dt\, dy$$

Computed by the recursive formula (integration by parts)

$$I_{\alpha,\beta}(z) = \frac{1}{\beta}\Big(z^{\alpha-1}\, e^{-z\,\beta} + (\alpha-1)\, I_{\alpha-1,\beta}(z)\Big)$$

Candidate adjustment coefficients are the positive R for which

$$\int_0^\infty e^{R\,y} \int_z^\infty e^{-\theta\,y}\, d\theta\, dy$$

is finite, that is all $R < z$.

Conclude: The supply of parametric models that are realistic and at the same time feasible is limited when we would like to introduce dependence between counts and amounts.

NON-PARAMETRIC MODELS BASED ON THE DIRICHLET PROCESS

$(P_1, \ldots, P_d)$ is Dirichlet distributed with parameter $(\alpha_1, \ldots, \alpha_d) \in (0, \infty)^d$, written

$$(P_1, \ldots, P_d) \sim \mathrm{Dir}_d(\alpha_1, \ldots, \alpha_d)$$

if it has density

$$\frac{\Gamma\left(\sum_{i=1}^d \alpha_i\right)}{\prod_{i=1}^d \Gamma(\alpha_i)} \prod_{i=1}^d p_i^{\alpha_i - 1} \, dp_i$$

$(p_1, \ldots, p_d) \in [0, 1]^d$, $\sum_{i=1}^d p_i = 1$. The expected value of P_j is

$$\mathbb{E}\, P_j = \frac{\alpha_j}{\sum_{i=1}^d \alpha_i}$$

$(\mathbb{A}, \mathcal{A}, \alpha)$ is a measure space, α finite. A random measure P on $(\mathbb{A}, \mathcal{A})$ is said to be a Dirichlet process with parameter α,

$$P \sim \mathsf{Dir}(\alpha),$$

if for any measurable partition $(A_1, \ldots, A_d)$ of $\mathbb{A}$ we have

$$(P(A_1), \ldots, P(A_d)) \sim \mathsf{Dir}_d(\alpha(A_1), \ldots, \alpha(A_d))\,.$$

Let Y be a random element with values in $\mathbb{A}$ such that

$$Y|_P \sim P$$

The marginal distribution of Y is obtained by norming α to a probability measure

$$Y \sim P_0 = \frac{1}{\alpha(\mathbb{A})}\,\alpha\,, \quad \mathbb{E}[h(Y)] = \int h(y)\,P_0(dy)$$

$$(Y_1, \ldots, Y_n)|_P \sim P \times \cdots \times P$$

$$P|_{(Y_1,\ldots,Y_n)} \sim \text{Dir}\left(\sum_{i=1}^{n} \varepsilon_{Y_i} + \alpha\right)$$

Thus, the Dirichlet process is a natural conjugate prior of the non-parametric class of distributions on $(\mathbb{A}, \mathcal{A})$.
Denoting posterior distribution by P_n, we can rewrite

$$P_n = z_n \, \widehat{P}_n + (1 - z_n) \, P_0$$

$$\widehat{P}_n = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_{Y_i}, \;\; z_n = \frac{n}{n + \alpha(\mathbb{A})}$$

$$\mathbb{E}[h(Y)|Y_1, \ldots, Y_n] = z_n \int h(y) \, \widehat{P}_n(dy) + (1 - z_n) \int h(y) \, P_0(dy)$$

COUNTS INDEPENDENT OF AMOUNTS
THE LATTER INTERDEPENDENT

The time change technique used in Theorem 1 gives:
$\psi(u)$ with adjusted premium is the same as for λ constant.
Therefore, consider constant λ and dependent $Y_1, Y_2, \ldots$.

G is random distribution on $(\mathbb{R}_+, \mathcal{B}_+)$
Conditional on G, $Y_1, Y_2, \ldots$ are i.i.d. replicates of $Y \sim G$

$$\nu^N(dt, dy) = \lambda\, dt\, G_t^N(dy)\,, \;\; G_t^N = \mathbb{E}[G|N_t, Y_1, \ldots, Y_{N_t}]$$

By independence, consider N_t as fixed and work with $G_t^N = \mathbb{E}[G|Y_1, \ldots, Y_{N_t}]$.

$$m_{G_t^N}(R) = \mathbb{E}[e^{R\,Y}|Y_1, \ldots, Y_{N_t}]$$

$G \sim \text{Dir}(\alpha)$, where α has finite exponential moments.

$$c_t = \frac{\lambda}{R} \frac{\sum_{i=1}^{N_t} \left(e^{RY_i} - 1\right) + \int_0^\infty \left(e^{Ry} - 1\right) \alpha(dy)}{N_t + \alpha(\mathbb{R})}$$

c_t converges almost surely to the constant c for the classical model, now with G random but known in the limit.

DEPENDENT COUNTS AND AMOUNTS. SPARRE ANDERSEN

Dependence between counts and amounts complicates matters:
Both λ_t^N and G_t^N depend on $(N_s)_{s\in[0,t)}$ and $Y_1, \ldots, Y_{N_t}$
Can no longer consider N_t as fixed in calculation of G_t^N

Must look for feasible assumptions and be prepared to sacrifice information

Assume premium rate is adjusted only at claims epochs
Premium rate in $(T_{i-1}, T_i]$ by c_i is chosen at time T_{i-1}

Net loss in $(T_{i-1}, T_i]$ is $Y_i - c_i V_i$, where $V_i = T_i - T_{i-1}$

$$\psi(u) = \mathbb{P}\left[\bigcup_{n=1}^{\infty} X_{T_n} > u\right] = \mathbb{P}\left[\bigcup_{n=1}^{\infty} \sum_{i=1}^{n} (Y_i - c_i V_i) > u\right]$$

Can formally work in discrete time and denote

$$\mathcal{F}_n = \mathcal{F}_{T_n} = \sigma\{(Y_i, V_i);\, i = 1, \ldots, n\}$$

Introduce $M = (M_n)_{n=0,1,2,\ldots}$ defined by $M_0 = 1$ and

$$M_n = e^{R \sum_{i=1}^{n} (Y_i - c_i V_i)}, \quad n = 1, 2, \ldots$$

With R fixed by solvency considerations, adjust premiums to make M a martingale. This is obtained by determining c_{n+1} as the $\mathcal{F}_n$-measurable random variable that makes

$$\mathbb{E}\left[e^{R\,(Y_{n+1} - c_{n+1} V_{n+1})} \middle| \mathcal{F}_n \right] = 1. \tag{1}$$

Martingale argument in the proof of Theorem 2 carries over, showing that

$$\psi(u) \leq e^{-R\,u}$$

Specify feasible model:
$H \sim \text{Dir}(\alpha)$ on $(\mathbb{R}_+ \times \mathbb{R}_+, \mathcal{B}_+ \times \mathcal{B}_+)$
Conditional on H, (Y_i, V_i), $i = 1, 2, \ldots$ are i.i.d. replicates of a generic pair $(Y, V) \sim H$ (Sparre Andersen)
(1) becomes

$$\sum_{i=1}^{n} e^{R\,(Y_i - c_{n+1}\,V_i)} + \int\int e^{R\,(y - c_{n+1}\,v)}\,\alpha(dy, dv) \;=\; n + \alpha(\mathbb{R} \times \mathbb{R})$$

For exponential moments on the left hand side to exist, α needs to be such that

$$\mathbb{E}\, e^{R\,(Y - c\,V)} < \infty$$

for all $c > 0$ for some $R > 0$

Example.

$V \sim \text{Exp}(\lambda)$ and $Y|_V \sim \text{Exp}(\beta\, V)$:

$P_0(dy, dv) = k\, v\, e^{-\lambda v - \beta v y}\, dy\, dv$

$$\begin{aligned} \mathbb{E}\, e^{R\,(Y - c\,V)} &= k \int_0^\infty e^{Ry} \int_0^\infty v\, e^{-(Rc+\lambda+\beta y)\, v}\, dv\, dy \\ &= k \int_0^\infty \frac{e^{R\,y}}{(R\,c + \lambda + \beta\, y)^2}\, dy \end{aligned}$$

which is ∞ for all $c > 0$ and all $R > 0$. The scenario is not eligible.

Example.

$Y \sim \text{Exp}(\gamma)$ and $V|_Y \sim \text{Exp}(\lambda\, Y)$:

$P_0(dy, dv) = k\, y\, e^{-\lambda y v - \gamma y}\, dy\, dv$

$$\begin{aligned} \mathbb{E}\, e^{R\,(Y - c\,V)} &= k \int_0^\infty e^{-Rcv} \int_0^\infty y\, e^{-(\lambda v + \gamma - R)\, y}\, dy\, dv \\ &= k \int_0^\infty \frac{e^{-Rcv}}{(\lambda v + \gamma - R)^2}\, dv \end{aligned}$$

which is finite for all $c > 0$ and all $R \in (0, \gamma)$.

REFERENCES

Albrecher, H., Constantinescu, C., Loisel, S. (2011) Explicit ruin formulas for models with dependence among risks. *Insurance: Mathematics & Economics*, **48**(2), 265-270

Asmussen, S. (1999) On the ruin problem for some adapted premium rules. In *Probabilistic Analysis of Rare Events: Theory and Problems of Safety* (eds. Kalashnikov, V. and Andronov, A.M.), 1-19, Riga Aviations University. Downloadable at http://www.maphysto.dk/cgi-bin/gp.cgi?publ=77

Asmussen, S., Albrecher, H. (2010) *Ruin probabilities* 2nd edition, World Scientific, New Jersey

Bühlmann, H. (1972) Ruinwahrscheinlichkeit bei erfahrungstarifiertem Portfeuille. *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker* **72**, 211-224

Bühlmann, H. (2007) The history of ASTIN. *ASTIN Bulletin* **37**, 191-202

Bühlmann, H., Gerber, H.U. (1978) General jump processes and time change - or how to define stochastic operational time. *Scandinavian Actuarial Journal* **1978**, 102-107

Dubey, A. (1977) Probabilité de ruine lorsque le paramètre Poisson est ajusté a posteriori. *Bulletin de l'Association des Actuaires Suisses*, **2**, 211-224

Højgaard, B., Taksar, M. (1997) Optimal proportional reinsurance policies for diffusion models. *Scandinavian Actuarial Journal* **1997**, 166-180

Karr, A. (1991) *Point Processes and their Statistical Inference* 2nd edition, Marcel Dekker, Inc., New York, Basel, Hong Kong

Schmidli, H. (2008) Stochastic Control in Insurance. Springer-Verlag, London

MODELING AND MANAGEMENT OF EARTHQUAKE RISK IN INSURANCE

Joint work with Diego Jimenez-Huerta

Generic Earthquake: (T, L, M, Y):
$T \in (0, \infty)$ time of occurrence
$L \in (0, \ell)$ location (epicenter); $(x(L), y(L))$ latitude, longitude
$M \in (0, \infty)$ magnitude
$Y \in (0, \infty)$ insured loss

Indexed in chronological order, earthquakes form a marked point process (T_i, L_i, M_i, Y_i), $i = 1, 2, \ldots$. Associated counting measure is generated by

$$N(\mathcal{T}, \mathcal{L}, \mathcal{M}, \mathcal{Y}) = \sum_{i=1}^{\infty} \varepsilon_{(T_i, L_i, M_i, Y_i)}(\mathcal{T} \times \mathcal{L} \times \mathcal{Y} \times \mathcal{M}) = \int_{\mathcal{T}} \int_{\mathcal{L}} \int_{\mathcal{Y}} \int_{\mathcal{M}} N(dt, dl, dm, dy)$$

where ε_x denotes the measure with all mass 1 in x

Total insured loss in $(0, t]$: $X(t) = \int_0^t \int_{\mathcal{L}} \int_{\mathcal{M}} \int_{\mathcal{Y}} y N(dt, dl, dm, dy)$

THE CLAIM SIZE DISTRIBUTION

Insurance company offers comprehensive civil property insurance including coverage for earthquake risk. Individual policies are numbered by the chronology of their times of entry into the portfolio.
Policy record of policy No. $j = 1, \ldots, n$ specifies individual risk characteristics: time period $\mathcal{T}_j$ for which earthquake risk is covered, sum assured, self-insurance rule, technical description of insured object (building and contents), geographical coordinates (x_j, y_j) of the site of the insured property.
Statistical analysis of data from earthquake catalogue, policy records, and claims records, give the (estimated) conditional distribution of the claim amount caused by the impact of a given amount of seismic energy e on the property:

$$G_j(y|e)$$

Energy impact on risk No. j from generic earthquake (t, l, m) is

$$e_j(l, m) = \frac{m}{\sqrt{4/5((x_j - x(l))^2 + (y_j - y(l))^2)}}.$$

Losses incurred to different objects in an earthquake are conditionally independent given magnitude and epicenter. Distribution of the total loss due to earthquake (t, l, m) is convolution of the individual loss distributions $G_j(\cdot|e_j(l, m))$ for j such that $t \in \mathcal{T}_j$, $j_1(t), \ldots, j_{n(t)}(t)$ (say):

$$P_Y(dy|t, l, m) = G_1(\cdot|e_{j_1(t)}(l, m)) * \cdots * G_n(\cdot|e_{j_{n(t)}(t)}(l, m))\,(dy)$$

STRESS-RELEASE MODEL

Stress at time t

$$\begin{aligned} S(t) &= S(0) + a\,t - \sum_{i;T_i \le t} M_i \\ &= S(0) + a\,t - \int_0^t \int_0^\infty m\,N(d\tau, dm) \end{aligned}$$

Dynamics

$$dS(t) = a\,dt - \int_{m=0}^\infty m\,N(dt, dm)$$

Compensator (intensity measure)

$$\begin{aligned} &\mathbb{E}\left[N([t, t+dt), dl, dm, dy) \,|\, \mathcal{F}_{t-}\right] \\ &= \lambda(S(t-))\,dt\,P_L(dl|m, S(t-))\,P_M(dm|S(t-))\,P_Y(dy|l, m) \end{aligned}$$

MOMENTS OF TOTAL CLAIMS IN TIME $(t, u]$

$$V^{(k)}(t,s) = \mathbb{E}\left[(X(u) - X(t))^k \,|\, S(t) = s\right]$$
$$= \mathbb{E}\left[\sum_{j=0}^{k} \binom{k}{j} (X(t+h) - X(t))^j (X(u) - X(t+h))^{k-j} \,\middle|\, S(t) = s\right]$$
$$= (1 - \lambda(s)\, h)\, V^{(k)}(t+h, s+a\,h)$$
$$+\lambda(s)\, h \int\int\int \binom{k}{j} y^j V^{(k-j)}(t, s-m) P_L(dl|m,s) P_M(dm|s) P_Y(dy|t,l,m)$$

Insert

$$V^{(k)}(t+h, s+a\,h) = V^{(k)}(t,s) + \frac{\partial}{\partial t} V^{(k)}(t,s)\, h + \frac{\partial}{\partial s} V^{(k)}(t,s)\, a\, h$$

$$\frac{\partial}{\partial t}V^{(k)}(t,s) \;=\; -\lambda(s)\,V^{(k)}(t,s) \;-\; a\,\frac{\partial}{\partial s}V^{(k)}(t,s)$$
$$+\lambda(s)\int\int\int\binom{k}{j}y^j V^{(k-j)}(t,s-m))P_L(dl|m,s)P_M(dm|s)P_Y(dy|t,l,m)$$

Terminal condition at time u:

$$V(u,s) \;=\; 0\,,\;\; s\in[0,S(t)+a\,(u-t)]\,.$$

Central moments of $X(u) - X(t)$, given $\mathcal{F}_t$:

$m^{(1)} = V^{(1)}(u, S(t))$

$m^{(2)} = V^{(2)}(u, S(t)) - (V^{(1)}(u, S(t)))^2$

$m^{(3)} = V^{(3)}(u, S(t)) - 3V^{(2)}(u, S(t))V^{(1)}(u, S(t)) + 2(V^{(1)}(u, S(t)))^3$

Upper ε fractile of total loss $X(u) - X(t)$ is

$$\approx m^{(1)} + c_{1-\varepsilon}\sqrt{m^{(2)}} + \frac{c_{1-\varepsilon}^2 - 1}{6}\frac{m^{(3)}}{m^{(2)}}$$

$c_{1-\varepsilon}$ is upper ε-fractile of standard normal distribution.

PARAMETER ESTIMATION

Estimation of individual conditional claim loss distributions $G_j(\,\cdot\,|e)$ go by standard regression techniques, possibly GLIM or non-linear.

Parametric seismic model:
$\lambda_{\boldsymbol{\alpha}}(s)$, $P_L(dl|m,s) = q_{\boldsymbol{\alpha}}(l|s,m)\,dl$, $P_M(dm|s) = p_{\boldsymbol{\alpha}}(m|s)\,dm$
$\boldsymbol{\alpha}$ is a parameter in open set in Euclidean space of finite dimension.

At time t, likelihood of observations (T_i, L_i, M_i), $i = 1, \ldots, N(t)$, is

$$\begin{aligned} L &= \exp\left(-\int_0^t \lambda_{\boldsymbol{\alpha}}(S(t-))\,dt\right) \\ &\quad \prod_{i=1}^{N(t)} \lambda_{\boldsymbol{\alpha}}(S(T_i-))\,p_{\boldsymbol{\alpha}}(M_i|S(t-))\,q_{\boldsymbol{\alpha}}(L_i|S(t-),M_i) \end{aligned}$$

$$\begin{aligned}\ln L &= -\int_0^\tau \lambda_{\boldsymbol{\alpha}}(S(t-))\,dt \\ &\quad + \sum_{i=1}^{N(\tau)} \left[\ln \lambda_{\boldsymbol{\alpha}}(S(T_i-)) + \ln p_{\boldsymbol{\alpha}}(M_i|S(t-)) + \ln q_{\boldsymbol{\alpha}}(L_i|S(t-), M_i)\right]\end{aligned}$$

Maximum likelihood estimator $\boldsymbol{\alpha}^*$ is solution to

$$\frac{\partial}{\partial \boldsymbol{\alpha}} \ln L|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}^*} = 0$$

Asymptotic properties

$$\boldsymbol{\alpha}^* \xrightarrow{d} \mathrm{N}(\boldsymbol{\alpha}, I(\boldsymbol{\alpha})^{-1})$$

$$I(\boldsymbol{\alpha}) = \mathbb{E}\left(-\frac{\partial^2}{\partial \boldsymbol{\alpha} \partial \boldsymbol{\alpha}'} \ln L\right)$$

Problem: S is not directly observable. $S(0)$ is unknown random variable generated by the stationary distribution of S. Let $S(0)$ be a component of the parameter $\boldsymbol{\alpha}$. Example:
$\lambda(S(t)) = \exp(b + c\,S(t)), \quad c \in \mathbb{R},\ d > 0.$
Absorb $c\,S(0)$ into b, introduce total stress release in $[0, t]$ (observable)

$$R(t) = \int_0^\infty m\, dN_\tau(dm)$$

reparametrize $\alpha_1 = d + c\,S(0)$, $\alpha_2 = c\,a$, $\alpha_3 = -c$, to arrive at

$$\lambda(S(t)) = \exp(\alpha_1 + \alpha_2\, t + \alpha_3\, R(t))$$

Work with parametric intensity function

$$\lambda_{\alpha_1,\alpha_2,\alpha_3}(r, t) = \exp(\alpha_1 + \alpha_2\, t + \alpha_3\, r)\,.$$

Assume S has a stationary distribution F. This means that, if $S(0) \sim F$, then $S(t) \sim F$ for all $t > 0$. Thus, assume $S(0) \sim F$.

Let H be a continuously differentiable function such that $\mathbb{E}\, H(S(t)) = \int_0^\infty H(s)\, dF(s)$ is finite. Condition on what happens in $(0, h)$:

$$
\begin{aligned}
&\mathbb{E}\left[H(S_h) \mid S(0) = s\right] \\
&= (1 - \lambda_s\, h)\, H(s + a\, h) \;+\; \lambda(s)\, h \int_0^s p(m|s)\, dm\, H(s - m) \;+\; o(h) \\
&= (1 - \lambda_s\, h)\, (H(s) + H'(s)\, a\, h) \;+\; \lambda_s\, h \int_0^s p(m|s)\, dm\, H(s - m) \;+\; o(h) \\
&= H(s) \;+\; H'(s)\, a\, h \;-\; \lambda_s\, h \int_0^s p(m|s)\, [H(s) - H(s - m)]\, dm \;+\; o(h)
\end{aligned}
$$

Integrate w.r.t. F, using

$$\int_0^\infty \mathbb{E}\,[H(S_h)|\,S(0)=s]\;dF(s) \;=\; \mathbb{E}\,\mathbb{E}\,[H(S_h)|\,S(0)] \;=\; \mathbb{E}\,H(S_h)$$

to obtain

$$\begin{aligned}\mathbb{E}\,H(S_h) \;&=\; \int_0^\infty H(s)\,dF(s) \;+\; \int_0^\infty H'(s)\,dF(s)\,a\,h \\ &\quad -\int_0^\infty \int_0^s \lambda_s\,p(m|s)\,[H(s)-H(s-m)]\,dm\,dF(s)\,h \;+\; o(h)\,.\end{aligned}$$

Term on the left cancels against first term on the right, and we arrive at

$$a\int_0^\infty H'(s)\,dF(s) \;=\; \int_0^\infty \int_0^s \lambda_s\,p(m|s)\,[H(s)-H(s-m)]\,dm\,dF(s)\,.$$

This holds for all H satisfying the stated conditions and, in principle, this determines F.

It suffices to consider $H(s) = e^{-\eta\, s}$ which produces the Laplace transform

$$\widehat{F}(\eta) = \int_0^\infty e^{-\eta\, s}\, dF(s)\,,$$

which determines F uniquely. We find

$$a\,\eta \int_0^\infty e^{-\eta\, s}\, dF(s) = \int_0^\infty \int_0^s \lambda_s\, p(m|s)\; [e^{\eta\, m} - 1]\; dm\, e^{-\eta\, s} dF(s) \qquad (2)$$

Differentiate w.r.t. η and set $\eta = 0$:

$$a = \int_0^\infty \int_0^s m\, \lambda_s\, p(m|s)\, dm\, dF(s)$$

In the stationary state, build-up of stress per time unit equals expected stress release per time unit.

Explicit expression for $\widehat{F}$ (or F) only for simple specifications of $\lambda(s)$ and $p(m|s)$. Example:
Intensity proportional to the current stress and magnitude uniformly distributed:

$$\lambda(s) = \lambda\, s\,, \quad p(m|s) = \frac{1}{s}\, 1_{(0,s)}(m)\,,$$

Then (2) becomes

$$\begin{aligned} a\,\eta\,\widehat{F}(\eta) &= \lambda \int_0^\infty \int_0^s (e^{\eta\, m} - 1)\; dm\, e^{-\eta\, s} dF(s) \\ &= \lambda \int_0^\infty \left(\frac{e^{\eta\, s} - 1}{\eta} - s \right) e^{-\eta\, s} dF(s) \\ &= \lambda \int_0^\infty \left(\frac{1 - e^{-\eta\, s}}{\eta} - s e^{-\eta\, s} \right) dF(s) \\ &= \lambda \left(\frac{1}{\eta} - \frac{1}{\eta}\, \widehat{F}(\eta) + \widehat{F}'(\eta) \right) \end{aligned}$$

$$\widehat{F}'(\eta) - \left(\frac{a}{\lambda}\eta + \frac{1}{\eta}\right)\widehat{F}(\eta) = -\frac{1}{\eta}.$$

Solution

$$\widehat{F}(\eta) = \eta\, e^{\frac{a}{2\lambda}\eta^2} \int_\eta^\infty \frac{1}{h^2}\, e^{-\frac{a}{2\lambda}h^2}\, dh\,.$$

Integration by parts gives

$$\begin{aligned}\widehat{F}(\eta) &= \eta\, e^{\frac{a}{2\lambda}\eta^2}\left(\frac{1}{\eta}e^{-\frac{a}{2\lambda}\eta^2} - \frac{a}{\lambda}\int_\eta^\infty e^{-\frac{a}{2\lambda}h^2}\, dh\right)\\ &= 1 - \eta\,\frac{a}{\lambda}\, e^{\frac{a}{2\lambda}\eta^2}\int_\eta^\infty e^{-\frac{a}{2\lambda}h^2}\, dh\,.\end{aligned}$$

(l'Hospital's rule shows $\widehat{F}(0) = 1$, as it should.)

Mean stationary stress, $\mathbb{E}\,S(t) = -\,\widehat{F}'(0)$. Differentiate

$$\widehat{F}'(\eta) = -\frac{a}{\lambda}\,e^{\frac{a}{2\,\lambda}\,\eta^2}\int_{\eta}^{\infty} e^{-\frac{a}{2\,\lambda}\,h^2}\,dh - \eta\,\frac{a}{\lambda}\,e^{\frac{a}{2\,\lambda}\,\eta^2}\frac{a}{\lambda}\,\eta\int_{\eta}^{\infty} e^{-\frac{a}{2\,\lambda}\,h^2}\,dh + \eta\,\frac{a}{\lambda}$$

and calculate

$$\widehat{F}'(0) = -\frac{a}{\lambda}\int_{0}^{\infty} e^{-\frac{a}{2\,\lambda}\,h^2}\,dh = -\frac{a}{2\,\lambda}\int_{-\infty}^{\infty} e^{-\frac{a}{2\,\lambda}\,h^2}\,dh = -\frac{a}{2\,\lambda}\sqrt{2\pi\frac{\lambda}{a}}$$

to arrive at

$$\mathbb{E}\,S(t) = \sqrt{\frac{\pi\,a}{2\,\lambda}}$$

POISSON MODEL:

$$\mathbb{E}\left[N([t,t+dt),dl,dm)|\,\mathcal{F}_{t-}\right] \;=\; \lambda\,dt\,P_L(dl|m)\,P_M(dm)$$

Moments of total claims in $(t,u]$

$$\frac{d}{dt}V^{(k)}(t) \;=\; -\lambda\,V^{(k)}(t)$$
$$+\lambda\int\int\int\binom{k}{j}y^jV^{(k-j)}(t)P_L(dl|m)P_M(dm)P_Y(dy|t,l,m)$$

Terminal condition at time u:

$$V(u) \;=\; 0\,.$$

BAYESIAN POISSON-GAMMA MODEL

$\Lambda(\mathcal{L}) \sim \mathrm{Ga}(\alpha(\mathcal{L}), \beta)$
independent of $\Lambda(\mathcal{L}')$ if $\mathcal{L} \cap \mathcal{L}' = \emptyset$

$N(\mathcal{T}, \mathcal{L})|_\Lambda \sim \mathrm{Po}\,(\mu(\mathcal{T})\,\Lambda(\mathcal{L}))$
independent of $N(\mathcal{T}', \mathcal{L}')$ if $(\mathcal{T}, \mathcal{L}) \cap (\mathcal{T}', \mathcal{L}') = \emptyset$
μ Lebesgue measure

Magnitude M independent of everything else, with distribution P_M

$\Lambda(\mathbb{L}) \sim \mathrm{Ga}(\alpha(\mathbb{L}), \beta)$ with density

$$\frac{\beta^{\alpha(\mathbb{L})}}{\Gamma(\alpha(\mathbb{L}))} \lambda^{\alpha(\mathbb{L})-1} e^{-\beta\lambda}, \quad \lambda > 0,$$

independent of random measure Δ defined by

$$\Delta(\mathcal{L}) = \frac{\Lambda(\mathcal{L})}{\Lambda(\mathbb{L})}$$

and

$$\Delta \sim \mathrm{Dir}(\alpha) :$$

For partition $(\mathcal{L}_1, \ldots, \mathcal{L}_k)$, $(\Delta(\mathcal{L}_1), \ldots, \Delta(\mathcal{L}_k))$ has density

$$\frac{\Gamma(\alpha(\mathbb{L}))}{\prod_{i=1}^{k} \Gamma(\alpha(\mathcal{L}_i))} \prod_{1=1}^{k} q_i^{\alpha(\mathcal{L}_i)-1}$$

$q_i > 0$, $\sum_{i=1}^{k} q_k = 1$

Posterior distribution, given (T_i, L_i), $i = 1, \ldots, N(t)$:

$$\Lambda(\mathcal{L})|_{\mathcal{F}_t^N} \sim \mathsf{Ga}(\alpha(\mathcal{L}) + N((0,t], \mathcal{L}), \beta + t)$$

or

$$\Lambda(\mathbb{L})|_{\mathcal{F}_t^N} \sim \mathsf{Ga}\left(\frac{\alpha(\mathbb{L}) + N(t)}{\beta + t}\right)$$

$$\Delta|_{\mathcal{F}_t^N} \sim \mathsf{Dir}(\alpha(\,\cdot\,) + N(t, \,\cdot\,))$$

Predictive distribution for $X = X(u) - X(t)$ as in prior distribution, only with updated parameters.

CONDITIONAL CENTRAL MOMENTS:

$$\begin{aligned} M_1(\Lambda) &= \mathbb{E}[X|\Lambda] = \int_t^u \int\int\int y P_Y(y|l,m) P_M(dm)\Lambda(dl)dt \\ &= (u-t)\int m_1(l)\Lambda(dl) \end{aligned}$$

$$\begin{aligned} M_2(\Lambda) &= \mathbb{V}\mathrm{ar}[X|\Lambda] = \int_t^u \int\int\int y^2 P_Y(y|l,m) P_M(dm)\Lambda(dl)dt \\ &= (u-t)\int m_2(l)\Lambda(dl) \end{aligned}$$

$$\begin{aligned} M_3(\Lambda) &= \mathbb{E}[(X - M_1(\Lambda))^3|\Lambda] = \int_t^u \int\int\int y^3 P_Y(y|l,m) P_M(dm)\Lambda(dl)dt \\ &= (u-t)\int m_3(l)\Lambda(dl) \end{aligned}$$

UNCONDITIONAL NON-CENTRAL MOMENTS:

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[M_1(\Lambda)] \\
\mathbb{E}[X^2] &= \mathbb{E}[M_2(\Lambda)] + \mathbb{E}[M_1^2(\Lambda)] \\
\mathbb{E}[X^3] &= \mathbb{E}[(X - M_1(\Lambda) + M_1(\Lambda)]^3 \\
&= \mathbb{E}[M_3(\Lambda)] + 3\mathbb{E}[M_2(\Lambda)\, M_1(\Lambda)] + \mathbb{E}[M_1^3(\Lambda)]
\end{aligned}$$

UNCONDITIONAL CENTRAL MOMENTS:

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[M_1(\Lambda)] \\
\mathbb{V}\mathrm{ar}[X] &= \mathbb{E}[X^2] - \mathbb{E}^2[X] \\
\mathbb{E}[(X - \mathbb{E}[X])^3] &= \mathbb{E}[X^3] - 3\,\mathbb{V}\mathrm{ar}[X]\,\mathbb{E}[X] - \mathbb{E}^3[X]
\end{aligned}$$

Calculating the moments amounts to calculating expressions of the forms:

$$\mathbb{E}\left[\int m(l)\Lambda(dl)\right] = \int m(l)\mathbb{E}[\Lambda(dl)] \tag{3}$$

$$\mathbb{E}\left[\int m(l)\Lambda(dl)\int m'(l)\Lambda(dl)\right] = \int\int m(l)\, m'(l')\mathbb{E}[\Lambda(dl)\Lambda(dl')] \tag{4}$$

$$\mathbb{E}\left[\int m(l)\Lambda(dl)\int m'(l)\Lambda(dl)\int m''(l)\Lambda(dl)\right]$$
$$= \int\int\int m(l)m'(l')m''(l'')\mathbb{E}[\Lambda(dl)\Lambda(dl')\Lambda(dl'')] \tag{5}$$

Using that $\Lambda(\mathcal{L})$ and $\Lambda(\mathcal{L}')$ are independent if $\mathcal{L}\cap\mathcal{L}' = \emptyset$ and

$$\mathbb{E}[\Lambda(\mathcal{L})^k] = \frac{(\alpha(\mathcal{L})+k-1)\cdots\alpha(\mathcal{L})}{\beta^k}\ , k = 1, 2, ...$$

we calculate

$$\mathbb{E}[\Lambda(dl)] = \beta^{-1}\,\alpha(dl)$$

$$\mathbb{E}[\Lambda(dl)\,\Lambda(dl')] = \begin{cases} \frac{\alpha(dl)}{\beta}\frac{\alpha(dl')}{\beta}, & l \neq l' \\ \frac{(\alpha(dl)+1)\,\alpha(dl)}{\beta^2}, & l = l' \end{cases} = \beta^{-2}[\alpha(dl)\,\alpha(dl') + \delta_{l=l'}\alpha(dl)]$$

$$\mathbb{E}[\Lambda(dl)\,\Lambda(dl')\,\Lambda(dl'')] = \begin{cases} \frac{\alpha(dl)}{\beta}\frac{\alpha(dl')}{\beta}\frac{\alpha(dl'')}{\beta}, & l \neq l' \neq l'' \neq l \\ \frac{(\alpha(dl)+1)\,\alpha(dl)}{\beta^2}\frac{\alpha(dl'')}{\beta}, & l = l' \neq l'' \\ \text{similar expression if} & l = l'' \neq l' \text{ or } l \neq l' = l'' \\ \frac{(\alpha(dl)+2)(\alpha(dl)+1)\alpha(dl)}{\beta^3}, & l = l' = l'' \end{cases}$$

$$= \beta^{-3}[\alpha(dl)\,\alpha(dl')\,\alpha(dl'') + \delta_{l=l'\neq l''}\alpha(dl)\,\alpha(dl'') + \delta_{l=l''\neq l'}\alpha(dl)\,\alpha(dl')$$
$$+\delta_{l\neq l'=l''}\alpha(dl)\,\alpha(dl'') + 3\delta_{l=l'=l''}\alpha(dl)\,\alpha(dl) + 2\delta_{l=l'=l''}\alpha(dl)]$$
$$= \beta^{-3}[\alpha(dl)\,\alpha(dl')\,\alpha(dl'') + \delta_{l=l'}\alpha(dl)\,\alpha(dl'') + \delta_{l=l''}\alpha(dl)\,\alpha(dl')$$
$$+\delta_{l'=l''}\alpha(dl)\,\alpha(dl'') + 2\delta_{l=l'=l''}\alpha(dl)]$$

Inserting these things into (3) - (5), we find (in compact notation)

$$\mathbb{E}\left[\int m(l)\,\Lambda(dl)\right] = \beta^{-1}\int m\,d\alpha$$

$$\mathbb{E}\left[\int m(l)\,\Lambda(dl)\int m'(l)\,\Lambda(dl)\right] = \beta^{-2}\left[\int m\,d\alpha\int m'\,d\alpha + \int m\,m'\,d\alpha\right]$$

$$\begin{aligned}\mathbb{E}\Big[\int m(l)\,\Lambda(dl)\int m'(l)\,\Lambda(dl)\int m''(l)\,\Lambda(dl)\Big] &= \beta^{-3}\Big[\int m\,d\alpha\int m'\,d\alpha\int m''\,d\alpha \\ &+\int mm'd\alpha\int m''d\alpha + \int m\,m''\,d\alpha\int m'\,d\alpha + \int m'm''d\alpha\int m\,d\alpha \\ &+2\int m\,m'\,m''\,d\alpha\Big]\end{aligned}$$

From the results above we gather, using posterior parameters $\alpha_t = \alpha(\mathcal{L}) + N((0,t]$, $\beta_t = \beta + t$:

$$\mathbb{E}[X] = \frac{u-t}{\beta} \int m_1 \, d\alpha_t$$

$$\mathbb{V}\mathrm{ar}[X] = \frac{u-t}{\beta} \int m_2 \, d\alpha_t + \left(\frac{u-t}{\beta}\right)^2 \int m_1^2 \, d\alpha_t$$

$$\mathbb{E}[(X - \mathbb{E}[X])^3] = \frac{u-t}{\beta} \int m_3 \, d\alpha_t + 3 \left(\frac{u-t}{\beta}\right)^2 \int m_1 m_2 \, d\alpha_t$$
$$+ 2 \left(\frac{u-t}{\beta}\right)^3 \int m_1^3 d\alpha_t$$